Systems and Lattices

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1. Introduction. The theory of ideal systems has a bearing on a number of topics like the theory of rings, differential and difference rings, lattice ordered groups and rings, monoids and lattices. We shall here briefly indicate some of its applications to lattice theory; in particular in connection with concepts such as modularity, distributivity, relative complementation, canonical equivalences, lattice morphisms and representation theorems. This represents just a sample of possible applications and it is reasonable to expect that a fair amount of basic lattice theory will eventually be subsumed under the theory of systems.

We refer to [1] and [2] for definitions concerning ideal systems and to [4] as well as other subsequent papers for a more full exposition of the present material. We here add the following definitions. A closure system (D,x) is said to be a closure system with finite intersections in case any finite intersection of (non-void) closed sets is non-void. If the continuity axiom for an ideal system (D,x) is replaced by the weaker assumption $A_x B_x \subset (AB_x)_x$ we shall speak of a weak ideal system. And if the continuity axiom is dropped altogether (but keeping all the other axioms of an ideal system, including the multiplicative ideal property) we shall say that (D,x) is a generalized ideal system. (For the relevance of the notion of a weak ideal system we note in passing that homogeneous ideals in a graded ring, differential ideals in a differential ring and monadic ideals in a monadic algebra all form weak ideal systems which are not in general ideal systems.) We shall use the term <u>system</u> as a common designation for these various types of closure systems and ideal systems. We also recall that the system (D,x) is termed <u>semi-additive</u> if $C \subset A_x + B_x \ (= (A_x \cup B_x)_x)$ implies the existence of a subset E of B_x such that $A_x + C = A_x + E$. An essential tool in the theory of systems are the canonical equivalences associated with the closed sets (x-ideals) in (D,x). We define these by putting $b \equiv c(A_x)$ whenever $A_x + \{b\} = A_x + \{c\}$ and we can in particular consider corresponding quotient systems, canonical maps onto quotient systems, isomorphism theorems, Chinese remainder theorems etc.

2. <u>Modularity and distributivity in lattices</u>. It is a pleasant fact that the two basic assumptions in the theory of ideal systems (the continuity axiom and the additivity axiom respectively) when specialized to lattices simply reduce to distributivity and modularity respectively. The family of all ideals (here termed l-ideals) in a lattice L form a generalized ideal system (L,l) treating the intersection in L as multiplication. This gives a system with finite intersections in the sense of the introduction and a direct application of [2] (Theorem 2) gives the following

<u>Theorem 1.</u> <u>The following conditions are equivalent</u>

1. L <u>is a modular lattice</u>
2. (L,l) <u>is an additive system</u>
3. (L,l) <u>is a semi-additive system</u>
4. <u>Any canonical map</u> $(L,l) \to (L/A_l,\bar{l})$ <u>is a closed map</u>

5. The canonical map ${}^{B_1}/{}_{A_1 \cap B_1} \rightarrow {}^{A_1+B_1}/{}_{A_1}$ is bijective

6. The Chinese remainder theorem holds for two canonical equivalences in (L,1)

7. Any two canonical equivalences in (L,1) are permutable

8. The product of two canonical equivalences in (L,1) is a canonical equivalence in (L,1)

In a similar way the various formulations of the continuity axiom give rise to

Theorem 2. The following conditions are equivalent

1. L is a distributive lattice

2. (L,1) is an ideal system

3. (L,1) is a weak ideal system

4. $A_1 : b$ is always an 1-ideal

5. Any canonical equivalence in (L,1) is a lattice congruence

6. The Chinese remainder theorem holds for three (or more) canonical equivalences in (L,1)

7. Ideal multiplication in (L,1) is associative (here assuming the existence of a greatest element in L)

8. The ideal lattice of (L,1) is distributive

9. Every 1-ideal in L is an intersection of prime 1-ideals

We note in particular the connection between modularity and distrubutivity which is afforded by the Chinese remainder theorem.

3. Lattice morphisms and the shadow functor. Lattice morphisms do not behave well in general. They are for instance in general not determined by their kernels in case of lattices with zero (least element). It seems, however, that one can obtain some new clues to the study of lattice morphisms via various morphism concepts for categories of systems in combination with the use of a simple and well-behaved functor which gives the link between lattice morphisms and morphisms of systems.

The various domains of applications of the theory of systems give rise to a kind of forgetful functors which we shall call shadow functors. These are functors from the category of rings, graded rings, differential rings, lattices, distrubutive lattices, lattice ordered groups etc into various categories of systems. We have for instance a natural shadow functor $\underline{\mathrm{Sh}}_\delta$ from the category of commutative differential rings into the category of ideal systems which takes a differential ring R into the ideal system (R,δ) which is induced on the multiplicative monoid of R by the radical differential ideals of R and which takes a differential ring homomorphism $\Phi : R_1 \to R_2$ into the induced morphism of ideal systems $\underline{\mathrm{Sh}}_\delta(\Phi) : (R_1,\delta_1) \to (R_2,\delta_2)$. We can then say that $\underline{\mathrm{Sh}}_\delta(R)$ and $\underline{\mathrm{Sh}}_\delta(\Phi)$ are the shadows (or δ-shadows) of R and Φ respectively.

In case of a general lattice L , the system (L,l) is only a generalized ideal system and we get a shadow functor $\underline{\mathrm{Sh}}_l$ from the category $\mathscr{L}$ of lattices into the category of generalized ideal systems. We have the following simple, but basic result

Theorem 3. The shadow functor $\underline{\mathrm{Sh}}_l$ is faithful and full.

This theorem assures that statements concerning morphisms in the category of generalized ideal systems will specialize to statements concerning usual lattice morphisms when applied to this case. (This is far from true when it comes to ring theory and the corresponding shadow functor $\underline{\mathrm{Sh}}_d$)

One way of avoiding the pathologies of general lattice morphisms is of course to consider more restrictive classes of morphisms. In [2] the <u>additive morphisms</u> were devised for obtaining a fundamental homomorphism theorem for systems. Applied to lattices with zero a lattice morphism Φ is additive if and only if $\Phi(c) < \Phi(a)$ implies the existence of an element $b \in \mathrm{Ker}\,\Phi$ such that $c < a \cup b$. Denoting the category of distributive lattices with zero and general lattice morphisms by $\mathcal{L}d_o$ the situation is as follows.

<u>Theorem 4</u>. <u>Let</u> L_1 <u>and</u> L_2 <u>be distributive lattices with zero and let</u> $\Phi : L_1 \to L_2$ <u>be a lattice morphism</u>. <u>If</u> Φ <u>is surjective we have a canonical lattice isomorphism</u> $L_2 \cong {}^{L_1}/_{\mathrm{Ker}\,\Phi}$ <u>if and only if</u> Φ <u>is additive</u>. <u>Any</u> $\Phi \in \mathrm{Hom}_{\mathcal{L}d_o}(L_1, L_2)$ <u>is additive if</u> L_1 <u>is relatively complemented</u>. <u>Conversely, if for given</u> L_1 <u>in</u> $\mathcal{L}d_o$ <u>any</u> $\Phi \in \mathrm{Hom}_{\mathcal{L}d_o}(L_1, L_2)$ <u>is additive then</u> L_1 <u>is relatively complemented</u>.

The proof is easy using a well-known theorem of Hashimoto [7].

4. <u>Representation theorems</u>. In the foregoing we have indicated how certain concepts in the theory of systems apply by specialization to lattice theory. We shall now give an example of a reverse procedure taking the representation theorems for distributive lattices and Boolean algebras and try to generalize them to the theory of systems.

Theorem 5. *Let* (D,x) *denote an ideal system with zero* 0 *for which the sum of x-ideals is completely distributive with respect to intersection of x-ideals within the ideal lattice of* (D,x) *. If* $\sqrt{0}$ *denotes the nilpotent radical of* 0 *then we have a canonical injective morphism*

$$({}^{D}/_{\sqrt{0}}, x_0) \longrightarrow \prod ({}^{D}/_{P_x^{(i)}}, x_i)$$

where x_0 *denotes the canonical ideal system in* ${}^{D}/_{\sqrt{0}}$, x_i *denotes the canonical ideal system in* ${}^{D}/_{P_x^{(i)}}$ *and the product sign indicates product in the category of ideal systems, here taken over all prime x-ideals* $P_x^{(i)}$ *in* (D,x) .

Using Theorem 3 this theorem gives in particular the familiar representation theorem for complete Boolean algebras as an algebra of sets when specialized to the case $x = 1$.

We shall next announce a theorem which comprises the usual topological representation theorems for Boolean rings and distributive lattices as special cases. One introduces the prime spectrum $\mathrm{Spec}(D,x)$ (or simply $\mathrm{Spec}\, D$) of an ideal system (D,x) in the usual way (see [1]) and proves that it is a quasi-compact topological space in case D has an identity (which we assume is the case here). If $\varphi : (D_1, x_1) \to (D_2, x_2)$ is a morphism of ideal systems we put $\mathrm{Spec}\,\varphi\,(P_{x_2}) = \varphi^{-1}(P_{x_2})$ and note that $\mathrm{Spec}\,\varphi$ is a continuous map from $\mathrm{Spec}\, D_2$ into $\mathrm{Spec}\, D_1$ such that the inverse image of an open quasi-compact set is quasi-compact, in notation $\mathrm{Spec}\,\varphi \in \mathrm{Hom}_{\mathrm{Comp}}(\mathrm{Spec}\, D_2, \mathrm{Spec}\, D_1)$. This defines Spec as a contravariant functor from the category of ideal systems into the category of prime spectra of such systems (always assuming the extra compactness condition on the continuous maps).

In complete analogy with the notion of a Bezout ring we say that an ideal system is a Bezout system if every finitely generated ideal is principal. A radical ideal system is an ideal system where every ideal is identical with its nilpotent radical. By the Boolean part of an ideal system (D,x) we shall understand the ideal system which is induced by x on the submonoid $B(D)$ consisting of all the idempotents of D. We shall denote this ideal system by $(B(D),x)$ and recall that an ideal in $B(D)$ is nothing but a set $A_x \cap B(D)$ where A_x is an x-ideal in D. For what follows it is essential that all the ideal systems under consideration are principal in the sense that $(a)_x = Da$.

Theorem 6. The contravariant functor Spec from the category of principal and radical Bezout systems into the category of prime spectra of such systems is full. More precisely the canonical map

$$\mathrm{Hom}((D_1,x_1),(B(D_2),x_2)) \longrightarrow \mathrm{Hom}_{\mathrm{Comp}}(\mathrm{Spec}\, D_2, \mathrm{Spec}\, D_1)$$

is a bijection.

It is an immediate consequence of Theorem 6 and Theorem 3 that a distributive lattice or a Boolean ring is uniquely determined by its prime spectrum. As far as rings are concerned it semms that no essentially bigger class of rings than that of Boolean rings has this uniqueness property. In any case a regular, non-Boolean ring is not uniquely determined by its prime spectrum within the class of regular rings since it has the same spectrum as its Boolean part (considered as a ring).

This raises the question of whether one can obtain a topological representation for a more comprehensive class of ideal systems by taking also the localizations into account, i.e. consider affine schemes of ideal systems. Such an attempt was made in [6], at least arriving at results which comprise those of [5]. The notions of direct limit and localization extend naturally to ideal systems, but the general theory of affine schemes of ideal systems is still in an unsatisfactory state.

References

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